Macroscopic and Microscopic Models of Traffic Flow Under Various Stresses

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Abstract

We explore first-principles models for traffic flow stemming from two base models: a macroscopic PDE model of traffic density and a microscopic ODE model of vehicle movement in traffic. We examine the impacts on traffic flow from lane closures and traffic stops on traffic density. We also present simulated observations of the impact on traffic flow from changes in individual driver behavior.

1 Background and Motivation

Traffic is a nuisance that plagues just about every person in modern day society. The average American commute time is nearly 30 minutes, which translates to nearly 130 hours spent in a car on average per year. Reducing traffic would help save time on a large scale and lead to a better work-life balance.

The motivation behind these models is to better understand how traffic behaves under different conditions and assumptions and what causes optimal traffic situations. This paper will investigate microscopic and macroscopic models for common traffic situations.

Our microscopic model is inspired by the ODE presented by Ho [1] which models traffic based on near-neighbor interactions. Similar work on these interactions was presented by [4]. Our macroscopic models draw form a standard conservation law PDE for traffic flow as presented in the notes the text for this course [3]. [2] also investigated both microscopic and macroscopic models with lane changes and lane closures.

2 Macroscopic Modeling

This section models the density of traffic as a continuous function of space and time. Throughout this section, we begin with the classic traffic density PDE inspired by a conservation law

$$u_t + \partial_x(uV) = f \tag{1}$$

where $u(x,t)$ refers to the density of traffic, $V(x,t)$ refers to the velocity of traffic, and $f(x,t)$ represents source terms (i.e., highway entrances or exits). We let $x \in [0, L]$ and $t \in [0, T]$ [3].

2.1 Lane Closure

We first turn to the impact on traffic flow when of temporarily closing one or more lanes of a highway. This kind of closure is a common occurrence when a highway is under construction. Rather than close the entire highway, it is preferable to keep it partially open while the closed portion is being reconstructed.

To model this scenario, we begin with the PDE (1). For the sake of simplicity, we assume that there are no entrances or exits, so $f(x,t) = 0$.

Furthermore, for our choice of velocity function, we choose $V(x,t) = V_\infty(1 - \frac{u}{u_\infty g(x)})$, where V_∞ is the maximum velocity, u_∞ is the maximum possible traffic density on the entire highway, and $g : [0, L] \to [0, 1]$ is a function indicating what percentage of the highway is open. Using this velocity function in the traffic density PDE gives:

$$u_t + V_\infty u_x - \frac{2V_\infty}{u_\infty g(x)} u_x u + \frac{V_\infty g'(x)}{u_\infty g(x)^2} u^2 = 0.$$

To simplify this PDE, we non-dimensionalize with $\tilde{u} = \frac{u}{u_\infty}$, $\tilde{t} = \frac{V_\infty}{L} t$, and $\tilde{x} = \frac{x}{L}$. Dropping the tildes for further simplicity yields the following PDE:

$$u_t + (1 - \frac{2u}{g(Lx)}) u_x = -\frac{g'(Lx)}{g(Lx)^2} u^2. \tag{2}$$

Before continuing to solve this PDE, we need to choose $g(x)$. A natural choice for g is a variant of the standard normal distribution. We let

$$g(x) = 1 - \frac{n - l}{n} e^{-(x-f)^4}$$

where n is the total number of lanes of the highway, l is the number of lanes open for traffic, and f is a horizontal shifting factor (see Figure 1).

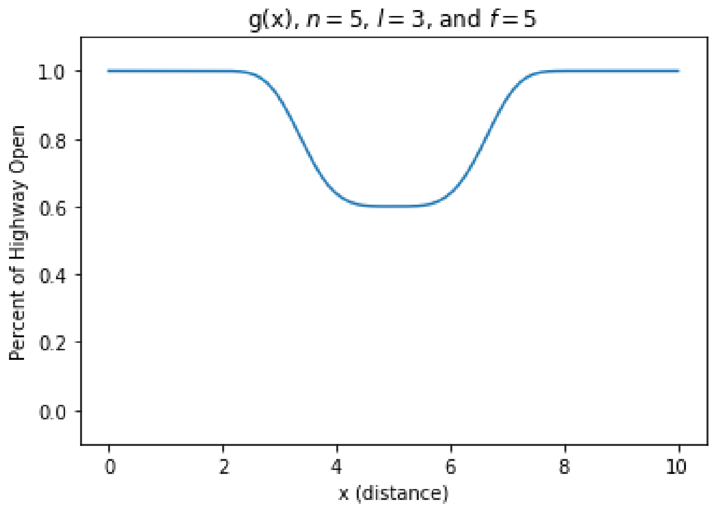

Figure 1: An example of the scaling function $g(x)$ for a temporary two-lane closure on a five-lane highway.

Lastly, we need to assume an initial density $u(x, t = 0)$. For simplicity, we let this be constant with respect to x, thus $u(x, t = 0) = c$. Using numerical solvers, we determined that for c big enough, the characteristics intersect. This means that there are shocks in the PDE, which result in a traffic jam.

2.1.1 Lane Closure Characteristics

We want to look at the characteristic curves of (2), which give

$$\frac{dt}{ds} = 1, \frac{dx}{ds} = 1 - \frac{2u}{g(Lx)}, \frac{du}{ds} = -\frac{g'(Lx)}{g(Lx)^2}u^2.$$

In order to solve these characteristic curves, we essentially need to solve a system of ODEs (which we will do numerically):

$$x_t = 1 - \frac{2u}{g(Lx)}, u_t = -\frac{g'(Lx)}{g(Lx)^2}u^2.$$

Figure 2 shows a set of possible characteristic curves for $u(x, t = 0)$ equal to 0, 0.3, and 0.5. The intersections of curves indicate a traffic jam. Naturally, for lower values of c (our initial density of vehicles) traffic jams are much less likely, and for $c = 0$ we see no shocks. This is shown in the left-most panel of Figure 2.

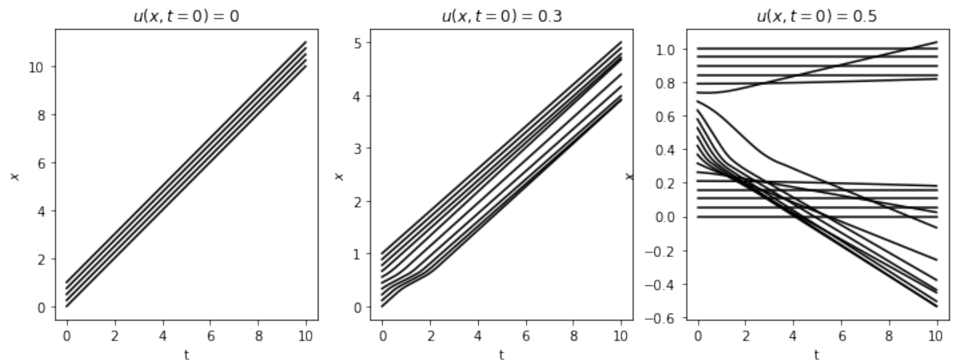

Figure 2: The characteristics for the Lane Closure scenario of a five lane road, with one lane temporarily closed. The fact that the characteristic curves intersect for a large enough c means that this scenario results in a traffic jam if the initial density is large enough.

2.2 After a Traffic Stop

We now turn to the behavior of traffic after traffic flow has stopped, such as what often happens after an accident on the freeway. At this point, we assume that we know the rate at which cars are leaving the zone we are interested in and we can analyze our PDE to find when the traffic density will reach an equilibrium point. For our purposes, we will begin by finding when the density will reach 0.

We again begin with the general conservation law traffic density PDE (1). We choose $f(x,t) = 0$ for simplicity, and we let our function $V(x,t)$ (velocity) be the canonical velocity as in [3]. That is,

$$V(x,t) = V_\infty(1 - \frac{u}{u_\infty}).$$

where V_∞ is the maximum velocity, u_∞ is the maximum possible traffic density on the entire highway, and $x \in [0, L]$ and $t \in [0, T]$ as before. This yields the full equation

$$u_t + u_x V_\infty - 2\frac{V_\infty u u_x}{u_\infty} = 0.$$

We now consider the following auxiliary conditions, by which we can analyze our PDE. Our initial condition, at time $t = 0$ is $u(x, 0) = u_\infty$ as there is no movement, and we are at maximum vehicle density at all points along the stretch of road. Some simple boundary conditions are $u_x(0, t) = 0$

and $u_x(L,t) = \alpha$ where α is a constant rate of vehicles leaving the area at point L.

2.2.1 Numerical Methods

We attempted a finite difference method to model this PDE. We started simply solving for u_x using an upwind method (because $V_\infty > 0$) as follows:

$$u_x = \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

If we recall that we can similarly write

$$u_t = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

then we can put all these together in our original PDE and we arrive at the formula

$$u(x,t+\Delta t) = u(x,t) - (\Delta t)V_\infty \left[2\frac{u(x,t)u_x(x,t)}{u_\infty} - u_x(x,t)\right]$$

where $t+\Delta t$ is our next time step and we replace $u(x,t)$ and $u_x(x,t)$ with their respective solutions found earlier in the time step loop. This is based on a simple forward Euler time-stepping method. For robustness, we then expanded this to use the trapezoid rule and Newton's method to account for the implicit term. This resulted in the update function:

$$\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + \frac{1}{2}(\Delta t)\left[f(u(x,t)) - f(u(x,t+\Delta t)\right] \\
&= u(x,t) + \frac{V_\infty(\Delta t)}{2}\left[u_x(x,t)\left(\frac{2u(x,t)}{u_\infty} - 1\right)\right. \\
&\qquad \left. + u_x(x,t+\Delta t)\left(1 - \frac{2u(x,t+\Delta t)}{u_\infty}\right)\right] \\
&= u(x,t) + \frac{V_\infty(\Delta t)}{2(\Delta x)}\left[(u(x+\Delta x,t) - u(x,t))\left(\frac{2u(x,t)}{u_\infty} - 1\right)\right. \\
&\qquad \left. + (u(x+\Delta x,t+\Delta t) - u(x,t+\Delta t))\left(1 - \frac{2u(x,t+\Delta t)}{u_\infty}\right)\right]
\end{aligned}$$

To account for the two Neumann boundary conditions, we approximate the derivative using centered difference at those points as follows:

$$0 = u_x(0,t) = \frac{u(\Delta x,t) - u(-\Delta x,t)}{2\Delta x} \implies u(-\Delta x,t) = u(\Delta x,t)$$

and

$$\alpha = u_x(L,t) = \frac{u(L+\Delta x, t) - u(L-\Delta x, t)}{2\Delta x}$$
$$\implies u(L+\Delta x, t) = 2\alpha(\Delta x) + u(L-\Delta x, t)$$

We then plug those back into our original Forward Euler equation to get approximations for $u(0,t)$ and $u(L,t)$. The code of this scheme can be found in Appendix A.

2.2.2 Numeric Method Results

We used the code in Appendix A.1 to run our numeric scheme. We chose $V_\infty = 80$ because that's close to a normal freeway speed limit. $u_\infty = 10$ was chosen simply for scale.

Our choice of u_∞ appears to relate with our choice for $\alpha = 1$ in that if we make u_∞ smaller, it simply speeds up the rate of $u \to \frac{u_\infty}{2}$. This could physically represent having more or less lanes of traffic, allowing more cars to be backed up at a time. It is interesting to see that the solution tends towards $\frac{u_\infty}{2}$ instead of 0. This could simply mean that we have an equilibrium there, and that no matter what, we will reach that equilibrium. Figure 3 shows our solution at a number of different time steps.

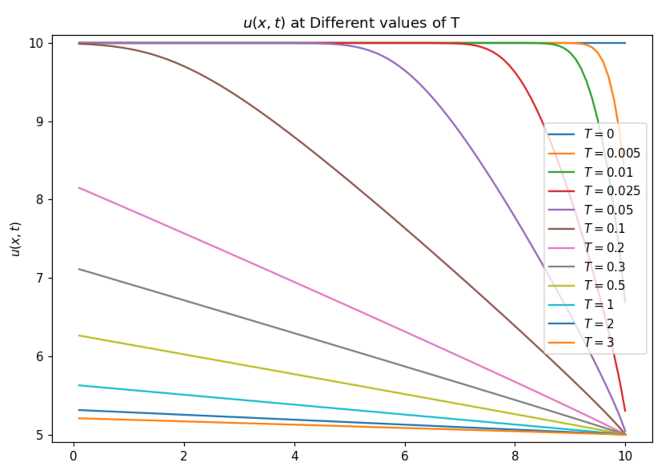

Figure 3: Numerically found solution for u(x,t) with boundary conditions $u_x(0,t) = u_\infty$ and $u_x(L,t) = \alpha$ where $\alpha = 1$. The solution is shown at a number of different time steps to show evolution of the density of traffic after a number of different time steps.

3 Microscopic Modeling

The approach in this section is to look at the dynamics of N cars on a length-L circuit, where the velocity and position of each car is considered individually. The cars are travelling in a single lane and cannot pass each other.

3.1 Baseline Model

Based on the work of Tiffany Ho [1], originally done in MatLab, we created a Python implementation of the traffic flow ODE

$$\ddot{x}_n = a[V(\Delta x_n) - \dot{x}_n]$$

where $n = 1, \ldots, N$, $\Delta x_n = x_{n+1} - x_n$, a is the sensitivity of the drivers (equal to 1 by default), and $V(\Delta x) = \tanh(\Delta x - C) + \tanh(C)$ is a nonlinear function representing velocity as a function of the distance between two cars, x_{n+1} and x_n. The C parameter manipulates the velocity function and can be interpreted as a driver "cautiousness" level. A low C level cause drivers to tailgate more, while a high C level leads drivers to decelerate faster when the distance from the proceeding car shrinks.

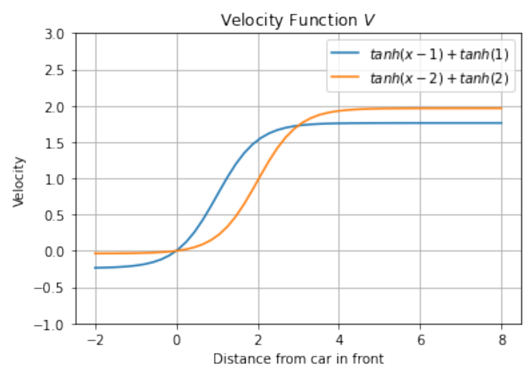

Figure 4: The realistic velocity functions, $V(x) = \tanh(x - C) + \tanh(C)$., for two different values of C. The C in both places is a measure of drivers' concern for safety. Raising it means cars will drive more slowly when the distance between the proceeding car gets smaller. It can range from 1 to 2.5 and still give meaningful results.

This system of N second-order differential equations can be reformulated into a system of $2N$ first-order differential equations through the following scheme:

For $i = 1, \ldots, N$, let

$$z_i = x_i$$

$$z_{N+i} = \dot{x}_i$$

Taking the time derivative of z_i for $i = 1, \ldots, 2N$, we get

$$\dot{z}_i = \begin{cases} z_{N+i} & \text{if } i \in \{1, \ldots, N\} \\ a[V(\Delta z_{i-N}) - \dot{z}_i] & \text{if } i \in \{N+1, \ldots, 2N\} \end{cases} \tag{3}$$

This is the system of $2N$ first-order equations that we solve numerically using the 4th Order Runge-Kutta method with the initial conditions described in the next section.

3.2 Initial Conditions

The steady-state flow of system (3) can be described by the following initial conditions:

$$x_n^{(0)} = bn$$

where $b = N/L$ represents the constant spacing between two successive vehicles. In addition, this model contains the boundary condition that the first car is equivalent to the $N+1$ car, making the distance traveled by each vehicle periodic. One can imagine N cars driving in a circle with a circuit length L with no passing allowed.

For our initial experiment, we initialize the initial displacement of the first vehicle to be slightly greater than the other vehicles:

$$x_1(0) = x_1^{(0)} + 0.1$$

$$x_n(0) = x_n^{(0)}$$

$$\dot{x}_n^{(0)} = 0$$

We observe the case with $L = 200$, $N = 100$, which gives

$$b = N/L = 2$$

The initial results of the baseline simulation are depicted in Figure 5.

3.3 Results of Initial Model

We see in Figure 5 that with the initial condition of $b = 2$ and initializing a single vehicle to have a slightly offset positional displacement, the model solution exhibits "stop-and-go" traffic. In an additional experiment where we set $b = \frac{1}{3}$ and included the same offset initial displacement for the first vehicle, the circuit does not get congested.

This shows that there could be a bifurcation point where congestion begins depending on a certain car-to-track length ratio. We plan on doing a more in-depth bifurcation analysis of this system in further work.

3.4 Bifurcation Testing

We conducted some further experimentation to determine at what car-to-road ratio that traffic would become congested. Figure 6 shows the displacement of a single car over time for several different values of $b = N/L$ with a fixed circuit length of $L = 100$. It appears that stop-and-go traffic starts when the $b = 0.36$.

Figure 7 shows the mean total velocity (mean velocity over every car at every timestamp) for various values of b. We see a sharp dip in the total velocity when $b = 0.36$.

However, the bifurcation point of traffic congestion for this model is also very dependent on the cautiousness parameter. This parameter describes a driver's tolerance for driving at high velocities close behind the car in front the We can see in Figure 8 that the lower the cautiousness, the less susceptible the road is to congestion. However, one must consider that having less cautious drivers increases the risk of an accident, which is not taken into account in this model.

3.5 "Myopic" vs. "Look-ahead" Driving

In this section, we consider a change in the velocity function that factors in the effect of drivers looking ahead to see how far ahead the second car in front of them is. (If every car on the circuit does this, then two cars in sequence will not collide.) Of course, in real life, we would still also need to factor in the distance to the first car ahead, to avoid rear-ending another car, but in this simulation this isn't a problem.

We call the type of driving modeled by the original velocity function "myopic" driving, where a driver bases his or her speed entirely on the car immediately in front of them. The alternative, introduced in the previous

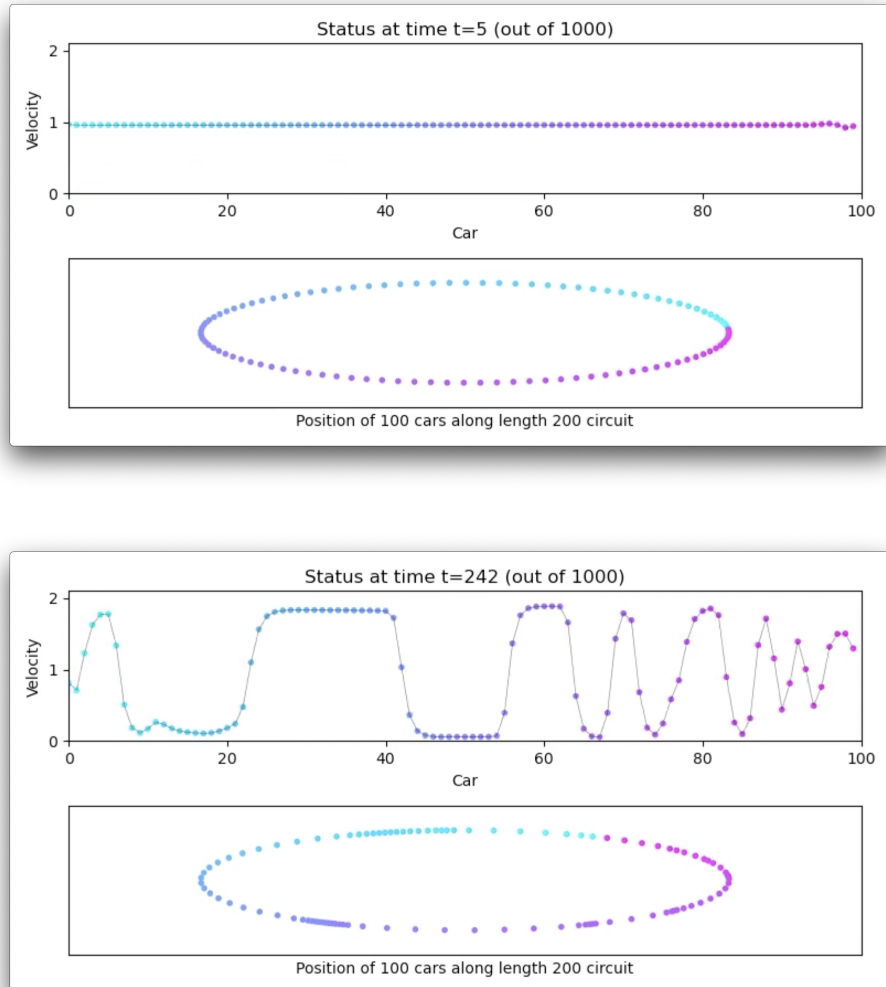

Figure 5: Example of Microscopic Model. It is clear that despite the smooth initial velocities for each car, patterns of stop-and-go traffic begin to form along the circuit as dense clusters of cars begin to form. Sections of traffic with low velocity are bunched closer together on the position plot, indicating traffic congestion.

paragraph, is "look-ahead" driving. The "Look-ahead" ODE becomes

$$\ddot{x}_n = a[V(\nabla x_n/2) - \dot{x}_n]$$

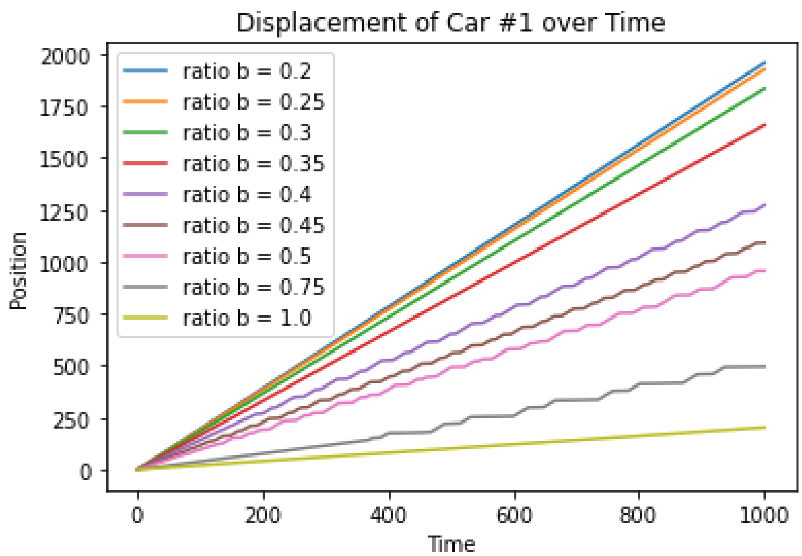

Figure 6: This shows the relationship between a single car's positional displacement over time for various values of the parameter $b = N/L$. The jagged lines shows evidence of stop and traffic for that value of b. This form of traffic congestion seems to start at around $b = 0.36$

where the authors have taken the mathematical liberty to define ∇x_n as $x_{n+2} - x_n$. The reason why we divide ∇x_n by 2 is because we don't want the doubling of distance between reference cars to mean doubling the velocity of the whole ODE.

With the necessary code adjusted for the tweak in boundary conditions this incurs, we are ready to compare the methods (see Figure 9).

While the two methods result in the same average velocity across the entire circuit, it is reasonable to suppose that look-ahead driving is superior because it requires less fuel consumption, as there are fewer changes in velocity. Thus, where drivers don't save on time, they save on money. Fewer changes in velocity also means less congestion. No driver will have to go through the trauma of driving at a speed slower than walking pace, but, in the end, the average velocity is no different between the two driving policies, so there is some sense of personal preference that can come into play.

We do have more reasons to favor look-ahead driving (as a driving policy), however, because of how it handles initial perturbations. In our microscopic ODE model in general, an initial perturbation refers to how much of

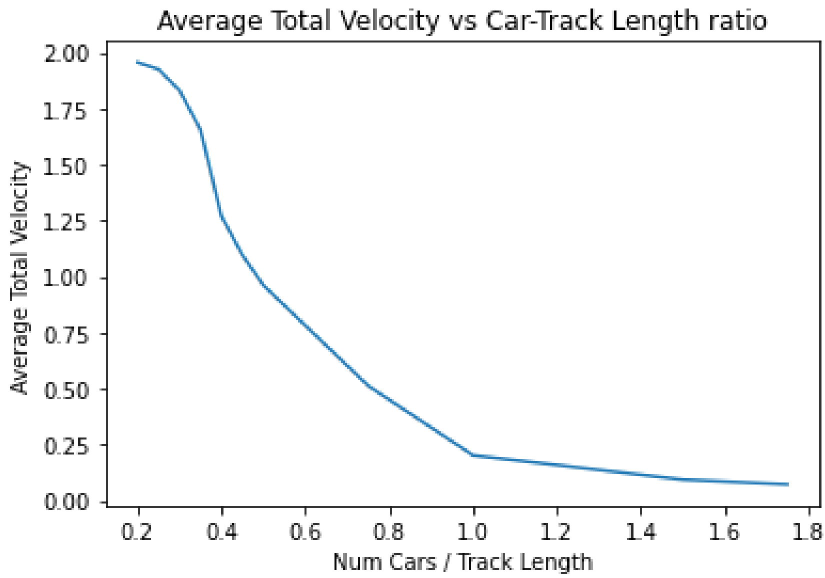

Figure 7: This shows how the total mean velocity of all the cars at every timestamp decreases the higher the Car Count-Track length ratio. The more cars, the slower everyone has to drive.

a position offset we give to the first car, defaulting to 0.1. If there were no offset, there would be no congestion or instability to analyze, which would be very boring. In Figure 10, we can see how the look-ahead driving model handles perturbations.

The effect of looking-ahead to the second car ahead has a damping effect on the traffic flow, suggesting that if drivers in the real world tried to pay attention to the second car ahead of them (and the first car in front of them as well, obviously), traffic would flow more smoothly. If average velocity of all cars on the circuit cannot be improved by adopting this behavior, then at least the traffic won't be stop-and-go this way.

3.6 Future Work

Future work could include exploring further modifications to the velocity equation V and modifications to the sensitivity parameter a, which was set to $a = 1$ and thus non-impactful for our experiments.

Another potential route would be a multi-lane derivation of this discrete model. While this model does great at simulating a single lane, both high-

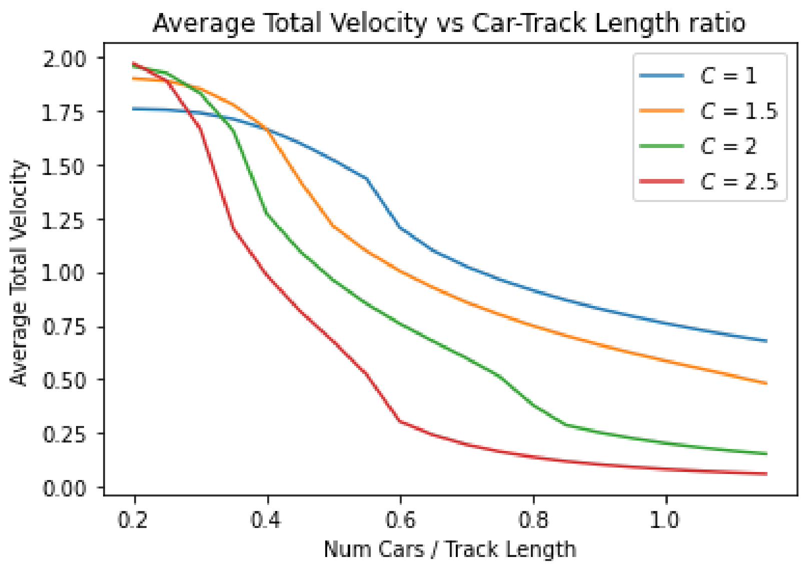

Figure 8: Evidence that the cautiousness of drivers has a big impact on traffic congestion. It can be seen that the more tailgating drivers, the less susceptible a road is to congestion given a high volume of cars. However, decreasing the cautiousness of drivers increases the risk of an accident.

way and city traffic scenarios involve a multi-lane dimension. It would be interesting to compare and contrast the microscopic and macroscopic models in terms of how they each describe a multi-lane traffic situation like a lane closure.

An approach we considered taking for simulating 2-lane traffic was to introduce a parallel circuit-ODE and allow the cars to change lanes (switch ODEs) when there is too much congestion in their current lane (as a function of velocity being too low for a certain number of time steps). This second inner lane would have length just less than the first lane (the radial difference in length will be taken into account). When a car changes lanes, its velocity will likely increase because there is a high chance that the car in front of it will be at a greater distance than next car in the past lane. It will be interesting to see if introducing the new lane will cause the density of traffic to diffuse evenly across the two lanes. This will depend on the parameters we use in deciding when a car would change lanes.

The approach we considered using for modeling multi-lane traffic experienced difficulties when it became less of an exercise in using ODEs and more

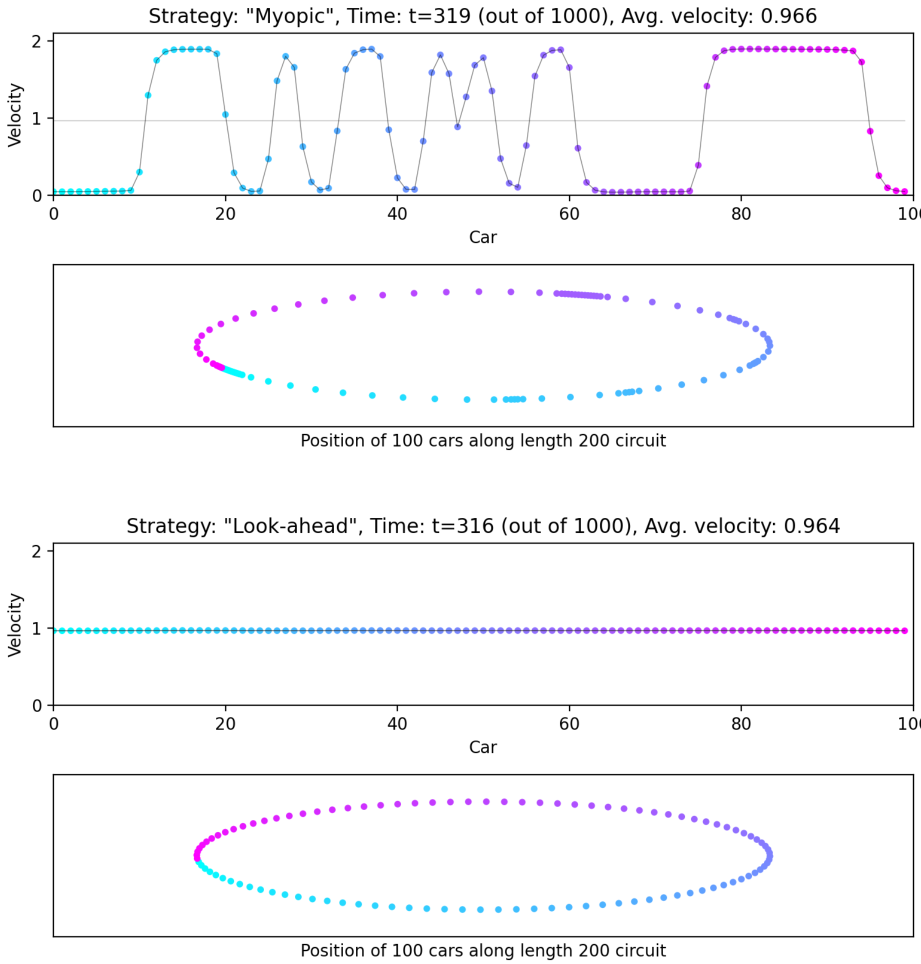

Figure 9: Myopic vs. Look-ahead driving. First, we note the obvious difference that exists between the two driving strategies: myopic driving results in very volatile, stop-and-go traffic patterns that have several dynamic pockets of congestion. Look-ahead driving smooths out the congestion into one egalitarian, flat traffic flow, where everyone equally suffers at the same velocity. Surprisingly, the average velocity is not different between the two strategies. While a myopic driving policy might be lobbied for by gas companies, it appears that look-ahead driving is better on fuel consumption from both the consumer's and environmentalist's perspective, and probably an emotional one, too.

of an exercise in coding up a dynamic simulation using clever coding, unrelated to ODEs. Because of the setup of the problem, changing the number of cars in a lane changes the dimensionality of the ODE for that lane, which is not easily handled by numerical methods, so we did not go forward with this effort.

4 Conclusion

In our macroscopic modeling, we found that our first-principles models produced the behavior we expected to see based on real life experience with traffic. We saw that lane closures caused shocks in the characteristics when the initial density was larger than a certain threshold. Traffic starting after a traffic stop began with high traffic density, slowly thinning out from the front to the back of traffic. The nonlinear nature is indicative of the observed "accordion" of cars after traffic stops. The consistent behavior of $u \to \frac{u_\infty}{2}$ specifically rather than 0 or some other value was somewhat unexpected and leaves room for further research and model adjustments.

In our microscopic modeling, we found that a large cause of traffic congestion is when drivers are not in sync and they are afraid of rear-ending each other, and when they only pay attention to the car directly in front of them. Behaviors that help dampen the stop-and-go patterns of traffic are when drivers showed greater awareness of other cars around them (in addition to the first car in front of them). While look-ahead driving helped prevent stop-and-go traffic (without improving average speed of traffic), interestingly, we found that lowering the concern for safety of drivers (making them more willing to drive at higher speeds when the car ahead of them is very close) also increased the average velocity for all cars.

Thus, our microscopic model suggests that the ideal driving policy is low-concern for safety combined with look-ahead driving. The two strategies combined smooth out traffic and increase overall average velocity. In other words, if drivers had a way to be aware of other drivers around them (beyond the car in front of them) and no fear of rear-ending while following the car in front of them closely, traffic would flow better. The shortcoming of this model, then, is how it fails to capture the fact that humans are unpredictable, and following behind too closely would result in many rear-ends. Also, since humans on the road are not perfectly connected with a hive-mind, they do not respond to changes in relative distances from each other as if they were an incompressible fluid. We do not recommend drivers lowering their concern for safety in the name of driving faster.

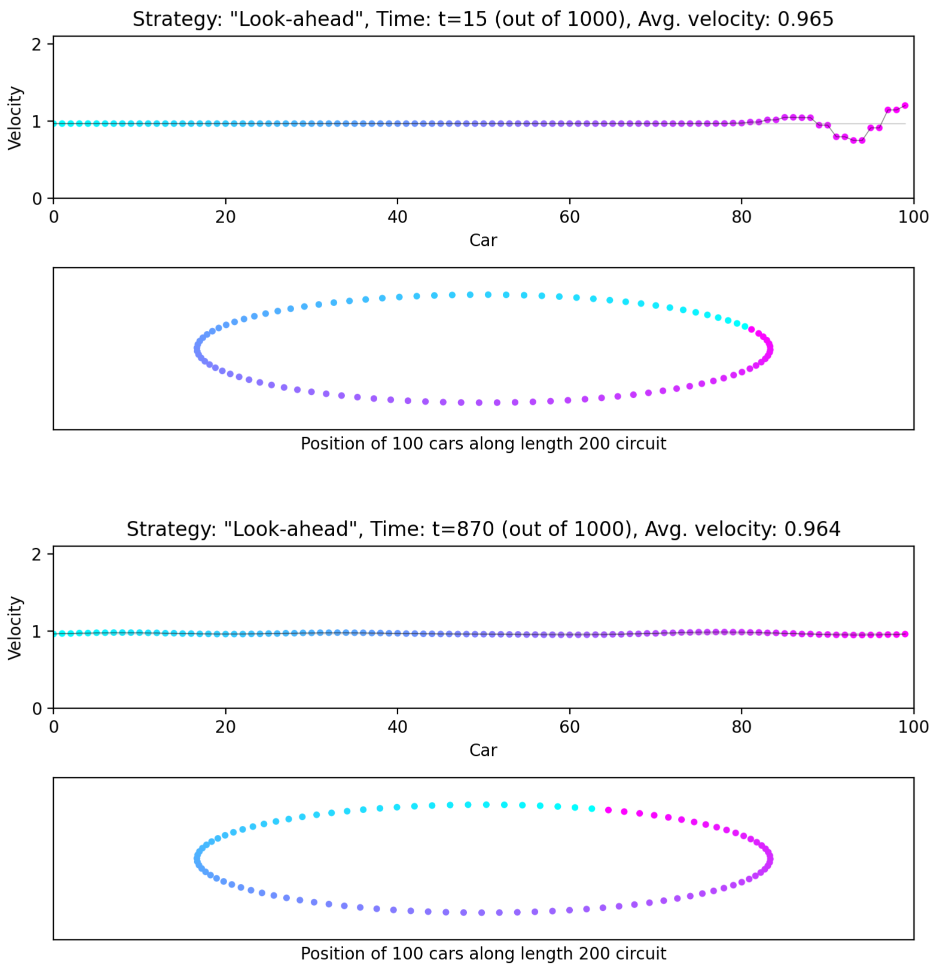

Figure 10: Look-ahead resistance to perturbations. The largest possible perturbation $p = .99$ immediately allows the cars before car 1 to speed up momentarily. Over time, the sinusoidal flow of velocity is dampened by the effect of look-ahead driving. No congestion forms. While never fully converging to a constant function, with long, slow, subtle oscillations still existing, the initial perturbation is hardly noticeable as time goes on. In myopic driving, larger initial perturbations create congestion quicker.

References

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A Code For the Homogeneous Macroscopic Model

```
import numpy as np
from scipy.optimize import newton

def traffic_density_trapezoid(V_inf, u_inf, dt, x, u0, u_xL, tol=1e-4, T=10):
    """
    Uses forward Euler and centered difference to
    solve the homogeneous macroscopic traffic
    density problem.

    Params:
      V_inf    : The maximum velocity
      u_inf    : The maximum density
      dt       : The time-step
      x        : The spatial discretization
      u0       : u(x, 0) - function
      u_x0     : u_x(0, t) - function (Neumann Boundary
                                   Condition)
      u_xL     : u_x(L, t) - function (Neumann Boundary
                                   Condition)
      tol      : The tolerance for being close enough
                   to 0
      T        : Time to go to, if tol not met first

    Returns:
      u : Our solution u(x, t)
      t : Time when u < tol
    """

    dx = x[1] - x[0]
    u = [u0(x)]

    def f(new_u, last_u, right):
        """
        Our objective function for Newton's method
        Params:
          new_u  : The current time step
          last_u : The previous time step
        """
        # Four parts of our update equation
        a = last_u[2:] - last_u[1:-1]
        b = (2*last_u[1:-1])/u_inf - 1
        c = np.append(new_u[1:], [right]) - new_u
        d = 1 - (2*new_u)/u_inf

        # Our update question set to `= 0`
```

```
        return new_u - last_u[1:-1] - (V_inf*dt)/(2*dx)*(a*b - c*d)

    t = dt
    while np.any(u[-1][1:] - .5*u_inf > tol) and t < T:
        # Get our initial guess (all ones)
        init_u = np.ones(len(u[-1]) - 2)
        # Get our right boundary condition
        uxL = 2*u_xL(t)*dx + u[-1][-2] - u[-1][-1]
        right = u[-1][-1] - (dt*V_inf)/(dx)*((2/u_inf)*u[-1][-1]*uxL - uxL)
        # Use Newton's method to solve for the next time step
        solved_u = newton(f, init_u, args=(u[-1], right))
        # Make sure our boundary conditions are satisfied
        new_u = np.concatenate((
                [u_inf],
                solved_u,
                [right]
            )
        )
        # Add the next time step's approximation to our result
        u.append(new_u.copy())
        t += dt
    return u, t
```

A.1 Code to Run the Function

```
V_inf = 80
u_inf = 10
dt = 1e-4
x = np.linspace(0, 10, 101)
u0 = np.vectorize(lambda x: u_inf)
alpha = 1
u_xL = lambda t: alpha
u, t = traffic_density_trapezoid(V_inf, u_inf, dt, x, u0, u_xL, T=3)
```

B Additional Code and Annimations

Further code for the numeric methods as well as animations of the models presented in this work can be found in the additional files submitted with this paper or by a request to the authors.